SOME GENERALIZATIONS OF WEAK CONVERGENCE RESULTS ON MULTIPLE CHANNEL QUEUES IN HEAVY TRAFFIC. I

H.A. Azarnoosh

Department of Statistics, Faculty of Science, Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran•

Abstract

This paper extends certain results of Iglehart and Whitt on multiple channel queues to the case where the inter-arrival times and service times are not necessarily identically distributed. It is shown that the weak convergence results in this case are exactly the same as those obtained by Iglehart and Whitt.

Introduction

The queuing systems considered here are the same as those which have been considered by Iglehart and Whitt [4] and [5], that is, they consist of r independent arrival channels and s independent service channels, whereas usually the arrival and service channels are independent. Arriving customers form a single queue and are served in the order of their arrival without defections. As in [4] and [5] we investigate two different models; the standard system and the modified system. The two models differ in their modes of operation for the service channels. In the standard system a waiting customer is assigned to the first available service channel and the servers (servers=service channels) are shut off when they are idle. Thus, the classical GI/G/s system is a special case of this system. In the modified system a waiting customer is assigned to the service channel that can complete his service first and the servers are not shut off when they are idle. For more details, see [4] and [5]. The modified system is of some interest in its own right and it has been introduced by Borovkov (1965) and used in [4] and [5] as an analytical tool.

Keywords: Weak convergence; Heavy traffic; Multiple channel queue

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Fairly extensive heavy traffic limit theorems for multiple channel queues have been obtained in [4] and [5]. We assume that the reader is familiar with these works. In [4] it was assumed that $r+s$ basic sequences of inter-arrival times and potential service times were independent sequences of non-negative, independent and identically distributed random variables with finite variance. In this present paper we relax the assumption that inter-arrival times and service times are identically distributed, i.e., we assume that $r+s$ basic sequences of inter-arrival times and service times are independent sequences of non-negative are independent random variables. We show that the weak convergence results in this case are exactly the same as those obtained in [4] and [5].

Preliminaries

The terminology and notations used throughout this paper are the same as in [4] and [5] and Billingsley (1968). We will also use notations from [2].

Conditions A:

(i) For each $n\geq 1$, let X^n_m, $m = 1,2, \ldots, k_n$ be mutually independent random variables defined on a probability space (Ω, B, P).

i)

$$X^n_m = o,\ var X^n_m = \sigma^2_{n,m},\ B^2_n = \sum_{m=i}^{kn} \sigma^2_{n,m},\ B^2_n/k_n \longrightarrow \sigma^2 \geq 1$$

as $n \longrightarrow \infty$ and $\max_{1 \leq k \leq k_n} \left| \sigma^2_{n,k} - \sigma^2_{n,k-1} \right| < 1/k^3_n$.

ii) $\frac{1}{B^2_n} \sum_{m=1}^{kn} \int_{\{|x| > \varepsilon B_n\}} x^2 dF_{n,m}(x) \longrightarrow 0$, as $n \longrightarrow \infty$

r all $\varepsilon > 0$,

here $F_{n,m}(x) = P\{X^n_m \leq x\}$, (Lindeberg condition).

Let $X^n_n(t)$, $t \in [0,1]$, be the random function defined in $D[0,1]$ by

$$X^n_n(t) = S^n_{[k_n t]}/B_n, \quad 0 \leq t \leq 1, \tag{1}$$

where $S^n_k = X^n_1 + ... + X^n_k$, $1 \leq k \leq k_n$, $S^n_0 = 0$.

Now, we state the following results without proof which we will use in the sequel.

Lemma 1. If the double sequence $\{X^n_m\}$, $n \geq 1$, $m = 1, ... ,k_n$, of r.v.'s satisfies condition A, then

$$\max_{1 \leq k \leq k_n} |X^n_k|/B_n \overset{P}{\to} 0,$$

([1], Lemma 2.2)

Throughout the paper Wiener processes will be denoted by ξ with and without subscripts or superscripts.

Theorem 1. ([1], Theorem 2.2) If the double sequence $\{X^n_m\}$ $n \geq 1$, $m = 1, ... , k_n$ of r.v.'s satisfies condition A, then $X^n_n \Rightarrow \xi$ in $D[0,1]$, where X^n_n is defined by (1).

It is clear that this theorem is an extension of theorem 16.1 of Billingsley (1968).

Condition B:

(i) For each fixed $n \geq 1$, let $\{u^n_m\}$ and $\{v^n_{m-1}\}$, $m = 1,2, ... ,k_n$, be two independent triangular sequences of independent non-negative random variables defined on some probability space (Ω, B, P). Thus for each $n \geq 1$, $X^n_m = v^n_{m-1} - u^n_m$, are independent for $m = 1, ... ,k_n$.

(ii) $E\,u^n_m = \lambda^{-1}_n \longrightarrow \lambda^{-1}$, $E\,v^n_m = \mu^{-1}_n \longrightarrow \mu^{-1}$ as $n \longrightarrow \infty$,

nd $Ex^n_m = \gamma_n = \mu^{-1}_n - \lambda^{-1}_n \to \gamma$ as $n \to \infty$ $m = 1, ... , k_n$

(iii) $Var(T) = \sigma^2_{n,m}(T)$, $\sum_{m=1}^{k_n} \sigma^2_{n,m}(T) \longrightarrow \sigma^2(T)$,

and

$$\max_{1 \leq m \leq k_n} \left| \sigma^2_{n,m}(T) - \sigma^2_{n,m-1}(T) \right| < 1/k^3_n,\ 0 < \sigma^2(T) < \infty,$$

where T stands for u or v.

(iv) The distribution of the random variables v^n_m and X^n_m satisfy the Lindeberg condition, that is,

$$\frac{1}{\sigma^2(v)\, k_n} \sum_{m=1}^{k_n} \int_{\{|y - \mu^{-1}| > \varepsilon\sqrt{k_n}\}} (y - \mu^{-1})^2\, dG_{n,m}(y) \longrightarrow 0$$

and

$$\frac{1}{\sigma^2 k_n} \sum_{m=1}^{k_n} \int_{\{|x-\gamma| > \varepsilon\sqrt{k_n}\}} (x - \gamma)^2 dF_{n,m}(x) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

for every $\varepsilon > 0$, where $F_{n,m}$ and $G_{n,m}$ are distribution functions of the random variables X^n_m and v^n_m respectively and

$\sigma^2 = \sigma^2(v) + \sigma^2(u)$.

For a double sequence $\{u^n_m\}$, $n \geq 1$, $m = 1, ... ,k_n$, of non-negative random variables we define a sequence of counting renewal process $\{A^n(t); t \geq 0\}$ for each $n \geq 1$, as follows

$$A^n(t) = \begin{cases} \max\left\{k \geq 1;\ \sum_{m=1}^{k} u^n_m \leq t\right\}, & u^n_1 \leq t \\ 0, & u^n_1 > t. \end{cases} \tag{2}$$

Let $Y^n_n(t)$, $0 \leq t \leq 1$, the random function induced by partial sums, be defined in $D[0,1]$ by

$$Y^n_n(t) = \frac{S^n_{[k_n t]} - [k_n t]\, \lambda_n}{\sigma(u)\, k^{1/2}_n},\ 0 \leq t \leq 1,$$

Where $S^n_k = u^n_1 + ... + u^n_k$, $1 \leq k \leq k_n$, and $S^n_0 = 0$.

If random variables u^n_m's satisfy condition B, then theorem 1 implies that $Y^n_n \Rightarrow \xi$ in $D[0,1]$. (3)

Define Z^n_n in $D[0,1]$ by

$$Z^n_n(t) = \frac{A^n(k_n t) - (k_n t)/\lambda}{\sigma_n \lambda^{-3/2} \sqrt{k_n}},\ 0 \leq t \leq 1,$$

where $A^n(t)$ is given in (2).

Now, if we proceed as theorem 17.3 of Billingsley (1968) by using Lemma 1 when r.v.'s satisfy condition

B, we have;

Theorem 2. $Z_n^n \Rightarrow \xi$ in $D[0,1]$, where Z_n^n is given in (3).

It is obvious that this theorem is an extension of theorem (17.3) of Billingsley (1968) and theorem 1 of Kyprianou (1971).

The Main results on Multiple Channel Queues in Heavy Traffic

(i) We adopt the same notations as those of Igelhart and Whitt's [4]. Assume that customers arrive one at a time in each of r channels and then immediately join a single queue in front of the s servers. So, we have as basic data $r + s$ independent sequences of non-negative and independent, but not identically distributed, random variables with finite mean and variance; $\{u_n^i;\ n\geq 1\}$ $(i=1,2,\ldots,r)$ and $\{v_n^j;\ n\geq 1\}$ $(j=1,2,\ldots,s)$ all defined on a common probability space (Ω,F,P). The variable u_n^i represents the inter-arrival time between the $(n-1)\,th$ and nth customers in the ith arrival channel and the variable v represents the nth potential service time of the jth server. We assume that the system is initially empty, although our limit theorem does not depend on this condition.

As in the previous part, we now define counting renewal processes associated with each channel as follows:

$$A^i(t) = \begin{cases} \max\{k \geq 1: u_1^i + u_2^i + \ldots + u_k^i \leq t\}, & u_1^i \leq t \\ 0, & u_1^i > t \end{cases}$$

for all $t\geq 0$, $1\leq i\leq r$, and,

$$S^j(t) = \begin{cases} \max\{k\geq 1: v_1^j + v_2^j + \ldots + u_k^j \leq t\}, & v_1^i \leq t \\ 0, & v_1^j > t, \end{cases}$$

for all $t\geq 0$, $1\leq j\leq s$.

It is clear that these processes represent the total number of arrivals or the total number of potential service times in the appropriate channel in the time interval (0,t].

Considering the service discipline in the modified system, it is very easy to express the queue length process, $Q'(t)$, in terms of these basic renewal processes. Throughout this paper all queue length processes count the customers being served as well as those waiting and there is no upper bound on the number of waiting customers.

For each $\omega \in \Omega$ and $t\geq 0$, we have

$$Q'(t) = X(t) - \inf\{X(s),\ 0\leq s\leq t\},$$

where $A(t) = A^1(t) + A^2(t) + \ldots + A^r(t)$,

$S(t) = S^1(t) + S^2(t) + \ldots + S^s(t)$,

and $X(t) = A(t) - S(t)$.

We assume that

$$\lambda_i = \frac{1}{Eu_1^i},\quad \mu_j = \frac{1}{Ev_1^i},\quad \lambda = \Sigma_{i=1}^r \lambda_i,\quad \mu = \Sigma_{j=1}^s \mu_j, \text{and}$$

$p = \lambda/\mu$. Furthermore, we let

$$\alpha_i^2 = \lambda_i^3 \sigma^2[u_1^i],\quad \sigma_i^2 = \mu_j^3 \sigma^2[v_1^j],\ \text{and}$$

$$\gamma^2 = \Sigma_{i=1}^r \sigma_i^2 + \Sigma_{j=1}^s \sigma_j^2$$

Now let $A_n^i (i=1,2,\ldots,r)$, $S_n^j (j=1,2,\ldots,s)$, X_n and Q_n' be random function in D[0,1] defined by

$$A_n^i \equiv [A^i(nt) - \lambda_i nt]/\alpha_i n^{1/2},$$

$$S_n^j \equiv [S^j(nt) - \mu_j nt]/\sigma_j n^{1/2},$$

$$X_n \equiv [X(t) - (\lambda-\mu)\,nt]/\gamma n^{1/2},$$

$$Q_n' \equiv [Q'(t) - (\lambda-\mu)\,nt]/\gamma n^{1/2},\quad 0\leq t\leq 1.$$

In this paper we will omit the proofs of those results which are based on the Iglehart and Whitt original ones [4]. In fact they are applicable in this case, of course, after necessary modifications. As we have mentioned in the previous part, ξ is a Wiener process, and random variables satisfy condition B.

Lemma 2. $X_n \Rightarrow \xi$.

The proof follows from lemma 2.1 of [4] by using theorems 1 and 2.

Now, let us introduce the continuous mapping $f: D \longrightarrow D$ which corresponds to an impenetrable barrier at the origin. For $x \in D$, f is defined by

$$f(x) = x(t) - \inf_{0\leq s\leq t} x(s),\quad 0\leq t\leq 1 \text{ and } x \in D.$$

(ii) **The Standard Queueing System**

In the standard multiple channel queueing system, as we have indicated in the introduction, customers are served in the order of their arrival by the first idle server.

The main idea, due to Borovkov ([3], section 5), is to define the standard system in terms of the same basic sequences of random variables already used for the modified system, and then show that the two queue length processes differ very little in heavy traffic. In other words, we show that the two corresponding sequences of random functions in D[0,1] converge to the same limit.

In order to investigate the standard system, we must generate the actual service times from the given sequences of potential service times. We follow the same procedure as Igelhart and Whitt's [4]. For each server we consider the actual service times to be a subsequence of the potential service times. If there is still a demand for service after a server has just served a customer, then let the next actual service time be the next random variable in the basic sequence of potential service times. If there has been no demand before receiving a customer at time t, let the next actual service time be the first unused random variable occurring after time t in the basic sequence of potential service times. In other words, the index of potential service time which is to be the actual service time of the next customer is $1 + max\{k, S^j(t)\}$, where k is the index of the potential service time which was the last actual service time. It is obvious that this procedure provides a subsequence of independent random variables.

We now let $Q(t)$, the queue length in the standard multiple channel queueing system, be the number of customers either waiting or being served at time t. Define the corresponding random function in D by

$$Q_n \equiv [Q(nt) - (\lambda - \mu)\, nt]/\gamma n^{1/2}, \quad 0 \le t \le 1.$$

The main result is:

Lemma 3. If $P=1$, then $Q_n \Rightarrow f(\xi)$; if $P>1$, then $Q_n \Rightarrow \xi$.

Proof: We proceed as theorem 3.1 of [4], using lemma 1 instead of lemma 3.3 of [4], this leads to the result.

(iii) The Departure Process

We denote the departure processes for the standard and modified queueing systems by $\{D(t); t \ge 0\}$ and $\{D'(t); \ge 0\}$, respectively. We define $D(t)$ $[D'(t)]$ to be the total number of customers which depart from the standard [modified] system in the interval (0,t]. Here we obtain weak convergence limit theorems for these processes when $P \ge 1$ as usual we assume $Q'(0)=Q(0)=0$, but justification for other initial conditions is theorem 4.1 of [2]. We have, by the definition of departure processes, $D(t)=A(t)-Q(t)$, and $D'(t)=A(t)-Q'(t)$. Now from the definition of $Q'(t)$, we have

$$D'(t) = A(t) - \{X(t) - \inf_{0 \le s \le t} X(s)\},$$

$$= S(t) + \inf_{0 \le s \le t} [A(s) - S(s)].$$

We now define the random function D'_n by

$$D'_n \equiv [D'(nt) - (\lambda \wedge \mu) nt]/n^{1/2}, \quad 0 \le t \le 1,$$

and continuous mapping g: D[0,1]xD[0,1] ⟶ D[0,1] by

$$g(x,y)(t) = y(t) + \inf_{0 \le s \le t} [x(s) - y(s)], \quad 0 \le t \le 1.$$

The process D_n is defined exactly like D'_n with $D(nt)$ replacing $D'(nt)$. Using lemma 3 we obtain the following lemma:

Lemma 4. If $P \ge 1$, then $d(D'_n, D_n) \Rightarrow 0$.

(iv) The Queue length Process at the ith Service Channel

Let $Q^i(t)$ be the number of customers in the standard system at time t which will be processed through the *ith* service channel and let $\{Q^i_n\}$ be the corresponding sequence of the random function in D where

$$Q^i_n \equiv Q^i(nt)/\gamma n^{1/2}, \quad 0 \le t \le 1.$$

Let the process $L^i(t)$ be the work load at time t which will be processed through the *ith* service channel. Define the corresponding sequence of random function $\{L^i_n\}$ in D by

$$L^i_n \equiv L^i(nt)/\gamma n^{1/2}, \quad 0 \le t \le 1.$$

Our goal is to show that $Q^i_n \Rightarrow (\mu_i/\mu) f(\xi)$ when $P=1$. To obtain this result, first we state the following lemmas:

Lemma 5. If $P \ge 1$, then $P(L^i_n, L^j_n) \Rightarrow 0, i,j=1,2,\ldots,s$. (cf. [4], lemma 5.1).

Lemma 6. If $P=1$, then $P(\mu_i^{-1} Q^i_n, \mu_j^{-1} Q^j_n) \Rightarrow 0$, $i,j=1,2,\ldots s$.

Proof. The result follows if we use an argument similar to that used in lemma 5.2 of [4], using theorem 1 instead of Donsker's theorem (cf. [2], theorem 16.1) and lemma 3.

Lemma 7. If $P=1$, then $P[(\mu_j/\mu) Q_n, Q^j_n] \Rightarrow 0$, $j=1,2,\ldots s$. (cf. [4], lemma 5.3).

(v) The Load and Waiting Time Processes

In the previous part we introduced the load at the *ith* service channel by $L^i(t)$. It is obvious that the total load for the entire system, $L(t)$, is just $L^1(t) + L^2(t) + ... + L^s(t)$. In a single server queue $L^i(t) \equiv L(t)$ is just the virtual waiting time, W(t), the time a potential customer arriving at time t would have to wait before reaching the server. Here

$$W(t) = \min_{1 \le i \le s} \{L^i(t)\}.$$

In this part our aim is to obtain the functional central limit theorems for $L^i(t)$, $L(t)$ and W(t) where $P=1$ and server are not identical.

The total load can be expressed as

$$L(t) = \sum_{j=1}^{s} \sum_{k=B^j(t)-[Q^j(t)-1]^+ +1}^{B^j(t)} v_k^j + \sum_{j=1}^{s} r_j(t),$$

where the v_k^j are actual service times, $B^j(t)$ is the total number of customers which arrive in (0,1] and are processed through the jth service channel, and finally $r_j(t)$ is the residual service time of the customer being served by the jth server at time t.

Define the random functions L_n and W_n in D[0,1] by

$$L_n \equiv L(nt)/\gamma n^{1/2}, \ 0 \le t \le 1,$$
$$W_n \equiv W(nt)/\gamma n^{1/2}, \ 0 \le t \le 1.$$

Now, we state the functional central limit theorems as follows:

Theorem 3. If $P=1$, then we have:

(a) $Q_n' \Rightarrow f(\xi)$ for all initial queue lengths,

(b) $D_n' \Rightarrow g(\alpha\xi_1, \sigma\xi_2)$ and $D_n \Rightarrow g(\alpha\xi_1, \sigma\xi_2)$,

(c) $Q_n^i \Rightarrow (\mu_i/\mu) f(\xi)$, $i=1,2,...,s$,

(d) $L_n \Rightarrow (s/\mu) f(\xi)$,

(e) $L_n^i \Rightarrow \mu^{-1} f(\xi)$ and $W_n \Rightarrow \mu^{-1} f(\xi)$,

the random function $f(\xi)$ has the same distribution as $|\xi|$.

Proof. (a) We proceed as theorem 2.1 of [4] by using lemma 2 instead of lemma 2.1 of [4].

(b) By applying the continuous mapping theorem and lemma 4 with the fact that $(\alpha A, \sigma S) \Rightarrow (\alpha\xi_1, \sigma\xi_2)$, the result follows.

(c) This follows by lemmas 3 and 5, and theorem 4.1 of [2].

(d) Using lemma 3 and theorem 4.1 of [2] it suffices to show that $d\left(L_n, \frac{s}{\mu} Q_n\right) \Rightarrow 0$. As it has been shown in the proof of lemma 5.2 of [2], the maximal residual service time can be omitted by the factor $n^{1/2}$, so we can ignore it. Thus, we can write,

$$\frac{L(nt)}{\gamma n^{1/2}} = \frac{1}{\gamma n^{1/2}} \sum_{j=1}^{s} \sum_{k=B^j(nt)-[Q^j(nt)-1]^+ +1}^{B^j(nt)} (N_k^j - \mu_j^{-1}) + \frac{1}{\gamma n^{1/2}} \sum_{j=1}^{s} [Q^j(nt)-1]^+ \mu_j^{-1}.$$

After dropping some terms of order $n^{-1/2}$, using triangle inequality we have,

$$d\left(L_n, \frac{s}{\mu} Q_n\right) \le p\left(L_n, \frac{s}{\mu} Q_n\right) \le p\left(\sum_{j=1}^{s} \mu_j^{-1} Q_n^j, \frac{s}{\mu} Q_n\right)$$

$$+ \sup_{0 \le t \le 1} \left| \frac{1}{\gamma n^{1/2}} \sum_{j=1}^{s} \sum_{k=B^j(nt)-[Q^j(nt)-1]^+ +1}^{B^j(nt)} (v_k^j - \mu_j^{-1}) \right|.$$

The first term on the right is less than or equal to

$$\sum_{j=1}^{s} \mu_j^{-1} p\left(Q_n^j, \frac{\mu_j}{\mu} Q_n\right)$$

and this converges in probability to 0 by lemma 7. Thus it will suffice to show that

$$\sup_{0 \le t \le 1} \left| \frac{1}{\gamma n^{1/2}} \sum_{j=1}^{s} \sum_{k=B^j(nt)-[Q^j(nt)-1]^+ +1}^{B^j(nt)} (v_k^j - \mu_j^{-1}) \right| \le$$

$$\sum_{j=1}^{s} \sup_{0 \le t \le 1} \left| \frac{1}{\gamma n^{1/2}} \sum_{k=B^j(nt)-[Q^j(nt)-1]^+ +1}^{B^j(nt)} (v_k^j - \mu_j^{-1}) \right| \Rightarrow 0.$$

This fact follows by an argument exactly like that used to prove lemma 5.2 of [4], and this leads to the result.

(e) (cf. [4], theorem 6.2).

Corollary 1. If $P=1$, then

$$\lim_{t \to \infty} P\left\{ \frac{Q'(t)}{\gamma t^{1/2}} \le x \right\} = \begin{cases} (2/\pi)^{1/2} \int_0^x \exp\{-y^2/2\} dy, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

Theorem 4. If $P>1$, then we have:

(a) $Q_n' \Rightarrow \xi$.

(b) $D_n' \Rightarrow \alpha\xi$ and $D_n \Rightarrow \sigma\xi$.

Proof. (a) Using lemma 2 and theorem 4.1 of [2] it suffices to show that $(X_n, Q_n^{'}) \Rightarrow 0$. The result follows by applying part (b) of theorem 2 of [5].

(b) By part (a) we have $d(X_n, Q_n^{'}) \Rightarrow 0$, it follows that $d(D_n^{'}, S_n) \Rightarrow 0$. Thus we have the result.

Corollary 2. If $P>1$, then

$$\lim_{t \to \infty} P\left\{ \frac{Q(t) - (\mu-\lambda)t}{\gamma t^{1/2}} \leq x \right\} = (1/\pi)^{1/2} \int_{-\infty}^{x} \exp\left\{-y^2/2\right\} dy.$$

It is clear that corollaries 1 and 2 also hold for Q(t).

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